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# Find the Relationship Between Two Triangles: Teachers solving a geometric problem using Dynamic Geometry software

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**Abstract**
This paper shows three ways a geometric problem could be solved using Dynamic Geometry software. Three high school geometry teachers who participated in a Summer Professional Development Institutes investigated a problem using dynamic geometry software (Dynamic Geometry software) to construct, explore, conjecture, and proof. The three teachers used different strategies both at the construction stage and writing the proof. Using the dynamic software enabled the teachers to conjecture and provided hints on how to tackle the proof. Although the three teachers had some experience with Dynamic Geometry software, the summer institute provided them with an environment to learn more about software usage.

*Keywords:* conjecture, construction, dynamic geometry, incorporating technology, Proof

## 1 Introduction

Integrating technology into mathematics instruction needs to be encouraged, because it has the potential to enhance teaching, and can help students access mathematical concepts. "Technology tools provide a powerful range of visual representation, which helps teachers to focus students' attention on mathematical concepts and techniques" (Koyuncu, Akyuz, & Cakiroglu, 2015, p. 837). Exposing teachers to technology use is an integral aspect of promoting teacher education (Kokol-Voljc, 2007).

Dynamic geometry such as a Dynamic Geometry software (Jackiw, 2001) has been in existence for some time and is believed to be a useful tool in the teaching and learning of geometry (Hollebrands, Laborde, & Straesser, 2008). The software is equipped with tools that enable one to construct, explore, and conjecture, and in some cases prove (Healy & Hoyles, 2002, p. 18). One of the useful features in dynamic software is dragging, which is "continuous real-time transformation. The dragging feature allows the user to change certain elements (e.g., a point) in a constructed geometric figure and observe the change of the corresponding geometric relationships in the figure" (An, & Nguyen, 2018, P. 45). The dragging feature and measurement tool in dynamic software allow the user to observe and investigate the constructed figure's measurement of sides or angles and change the figure being investigated to develop conjectures that can lead to proofs (Baccaglini-Frank, 2011).

## 2 Methods

Three high school teachers from low-income communities in a southern state of the United States were the participants of this qualitative case study. The Summer Institute staff chose, observed, and interviewed the three teachers who participated in the summer institute about their construction, their conjecture, and the proof they had produced. The three teachers were selected because of the distinctiveness of the proofs they gave. The answers of these three teachers are offered in order to demonstrate three distinct approaches to the same issue.

## 3 Results

This paper shows three different ways a geometric problem could be solved using Dynamic Geometry software by three high school mathematics teachers (all had some prior experience with Dynamic Geometry software) who were part of an NSF-funded Summer Institute. The Summer Institute was a ten-day Professional development seminar for the high school mathematics teachers who participated in the National Science Foundation grant project. The institute aimed to develop teachers' skills with Dynamic Geometry software software and associated classroom activity that were designed by the project personnel for use in their classroom for instruction. During one of the summer institute sessions; teachers were given the following problem to investigate:

Given ΔABC (Figure 1) is an arbitrary triangle such that Points D, E, and F are respectively on sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$.

$BD = \frac{1}{3}BC$, $\overline{CE} = \frac{1}{3}\overline{CA}$, and $\overline{AF} = \frac{1}{3}\overline{AB}$.

PQR is the triangle formed in the center by the three intersection points of the internal segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. What is the relationship between ΔPQR and ΔABC (Steinhaus, 1950)

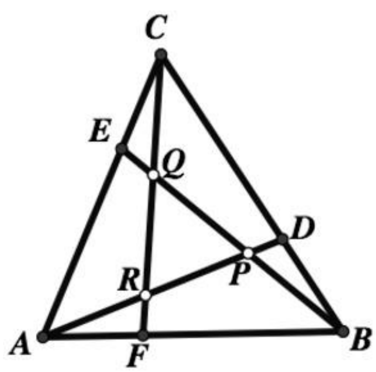


*Figure 1*: Steinhaus triangle

The aim of the task was to see how many different construction and proof strategies teachers use while using Dynamic Geometry software and to role-play the implementation of this problem in a classroom. This problem can be solved in multiple ways and provided the opportunity for teachers to learn important mathematical ideas and help teachers and students to think through important topics and how those topics might be understood. As NCTM (2000) points out,

> In effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually. Well-chosen tasks can pique students' curiosity and draw them into mathematics. … Regardless of the context, worthwhile tasks should be intriguing, with a level of challenge that invites speculation and hard work. Such tasks often can be approached in more than one way ... which makes the tasks accessible to students with varied prior knowledge and experience. (NCTM 2000, pp. 18–19)

Using different strategies, teachers constructed Figure 1.

Three teachers (we will call them Jane, John, and Mary) who were part of the Summer Institute were selected, observed and interviewed by the Summer Institute personnel about their construction, the conjecture they arrived at, and the proof they had written. The three teachers were chosen based on the uniqueness of the type of proof they provided. The solutions of these three teachers are presented to show three different ways to solve the problem.

### 3.1 Construct and Conjecture

Jane constructed Figure 1 by using calculator tools in Dynamic Geometry software, first by creating an arbitrary triangle ABC and measuring segment $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Then, using the calculator tool to find $\frac{1}{3}$ of segment $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ as referred to in the problem. She then constructed a circle. For example, to construct $\frac{1}{3}\overline{AB}$, she first selected point A as the center of the circle and selected $\frac{m\overline{AB}}{3} = 2.33$ as the radius to construct circle $c_1$ to meet segment $\overline{AB}$ at point F, which makes Segment AF equal to $\frac{1}{3}\overline{AB}$ (the same was done for segment $\overline{CA}$ and $\overline{BC}$ to create point E and D respectively (see Figure 2 below).

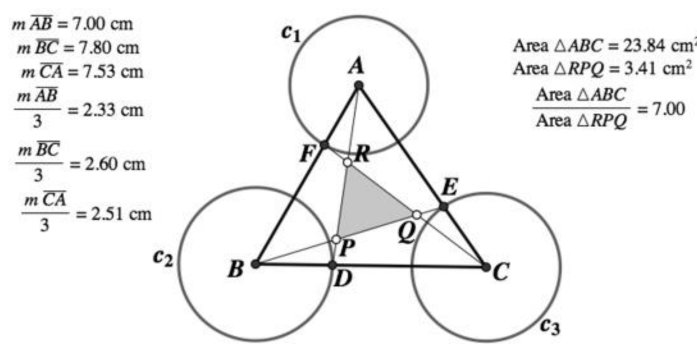


*Figure 2*: Jane's construction

When asked to make an educated guess about the relationship between ΔPQR and ΔABC, Jane suggested that ΔPQR is $\frac{1}{8}$ of ΔABC. Asked to explain why, Jane responded: "I divided the triangle into an approximately equal areas of ΔPQR and came up with eight regions, meaning that the relationship is $\frac{1}{8}$". It was not clear how Jane divided the region. She insisted that she did it in her head. Jane was then asked to investigate the relationship using Dynamic Geometry software. She then realised that ΔPQR is $\frac{1}{7}$ of ΔABC, which then became the conjecture that needed to be proven. Jane was so excited that she jumped out of her seat and shouted, “my educated guess was not correct but very close! and looking forward to proving the conjecture”.

John used the theorem which states that if parallel lines cut off congruent segments on one transversal, they then cut off congruent segments on any other transversals.

John started the construction by first constructing an arbitrary ΔABC. Then, from point A, he constructed a segment ( $\overline{AL}$ ) as shown in Figure 3. Using segment $\overline{HI}$ as the radius, he constructed three circles with centers at points A, J and K respectively. He then joined point L to point C. He constructed a parallel line to segment $\overline{LC}$ thorough point K to meet segment $\overline{AC}$ at point E. John continued the same for sides $\overline{BC}$ and $\overline{AB}$ as shown.

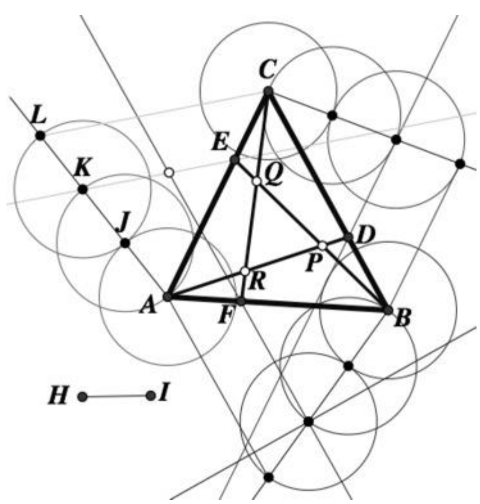


*Figure 3*: John’s construction

Asked to make an initial conjecture regarding the relationship between ΔPQR and ΔABC, John noted that ΔPQR is $\frac{1}{3}$ABC. When John was asked why $\frac{1}{3}$ ; he referenced $\frac{1}{3}$ ratio mentioned in the problem: “since the segments are in the ratio 1:3, the ratio of area of ΔPQR to ΔABC might be 1:3 also”. Using Dynamic Geometry software, John realised that his initial conjecture was wrong and that ΔPQR is $\frac{1}{7}$ΔABC, for which he later provided proof.

Mary's construction used the dilation tool in Dynamic Geometry software to construct $\frac{1}{3}$ of the segment. She first started with arbitrary ΔABC and “then double clicked point A," indicating the center of dilation, and then clicked point B. From the menu, she selected Dilate, after which the dilation window came up, and she then selected a fixed ratio. On the scale factor, she selected 1.0, and then input 3.0 for the next box (Figure 4) to create point F. Mary did the same for segment $\overline{CA}$ and $\overline{BC}$ to create points E and D respectively.

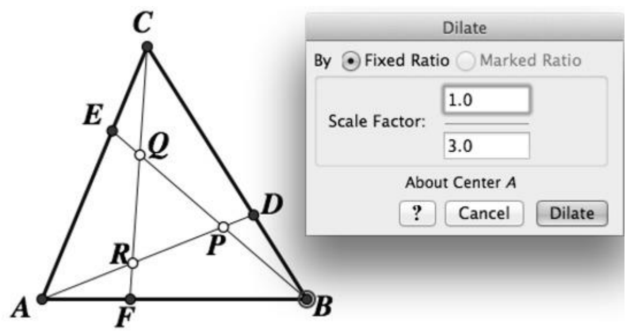


*Figure 4*: Mary’s construction

Mary was then asked to state the initial conjecture about the relationship between ΔPQR and ΔABC. She counted the number of regions created inside ΔABC, which were 7. She then noted that the relationship was $\frac{1}{7}$. Mary could not explain why she assumed that all seven areas had the same result and that the relationship was $\frac{1}{7}$. Asked to check the initial conjecture, Mary measured the area of both triangles ΔABC and ΔRQP, and of course, she was right, and confirmed that "it was an educated guess."

### 3.2 Solutions to the Problem

The three participants then moved on to prove the conjecture that area ΔPQR is $\frac{1}{7}$ of ABC as explained below:

*Jane's Solution: Concept of Similar Triangles*

Jane used the concept of similar triangles in working on her proof as seen in figure 5. She did that by creating a parallel line to create a similar triangle to enable her to compare triangles and thereby establish a relationship to aid her proof, as in figure 5 below. The creations of similar triangles were a result of her Dynamic Geometry software investigation.

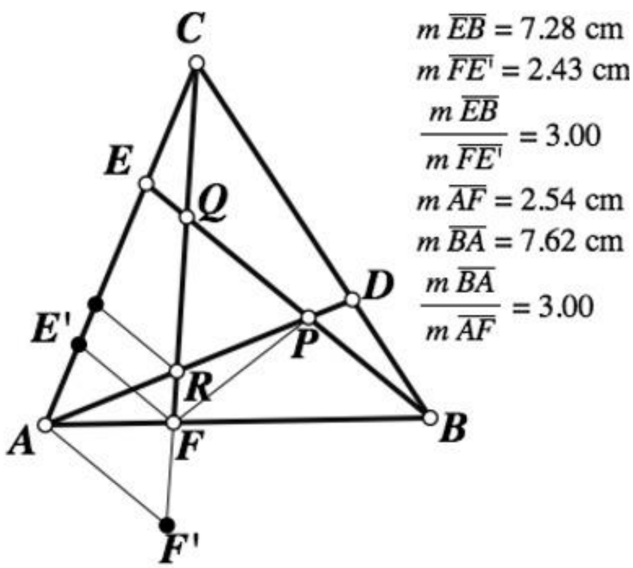


*Figure 5*: Jane's proof construction

Jane constructed a line parallel to the EB passing through point A and extended CF so that they intersected at the point F' (the line *AF'*). Next she constructed another line parallel line to EB through F (the line E'F).

Jane noted that because and $\overline{EB}||\overline{E'F}$(by construction), congruent corresponding angles are formed by a transversal cutting the parallel segments. Thus ∠AE'F ≅ ∠AEB and ∠EBA≅ ∠E'FA. Also, ∠A ≅ ∠A, so ΔAEB~ΔAE′F by AAA.

Jane explained that two triangles are similar if and only if the corresponding angles are congruent and corresponding sides are proportional and therefore:

$\frac{\overline{E'F}}{\overline{EB}} = \frac{\overline{AF}}{\overline{AB}} = \frac{\overline{AE'}}{\overline{AE}}$ (corresponding sides are proportional). She continued by noting that from construction of triangle ABC and points E, D and F (Ref Investigation: Given ΔABC (Figure 1) is an arbitrary triangle such that Points D, E, and F are respectively on sides.

$BD = \frac{1}{3}BC$, $\overline{CE} = \frac{1}{3}\overline{CA}$, and $\overline{AF} = \frac{1}{3}\overline{AB}$). Therefore;

$\frac{\overline{AF}}{\overline{AB}} = \frac{1}{3}$ by construction. Jane then concluded that since $\frac{\overline{E'F}}{\overline{EB}} = \frac{\overline{AF}}{\overline{AB}} = \frac{\overline{AE'}}{\overline{AE}}$, then

$\frac{\overline{E'F}}{\overline{EB}} = \frac{1}{3} = \frac{\overline{AF}}{\overline{AB}}$, and $\frac{\overline{AE'}}{\overline{AE}} = \frac{1}{3}$

Since $\frac{\overline{CE}}{\overline{CA}} = \frac{1}{3}$,

Assume AE' = 1, then AE = 3, $CE = \frac{AE}{2}$ = 1.5, EE′= 2, and CE′ = 2 + 1.5 = 3.5,

thus, $\frac{\overline{CE}}{\overline{CE'}} = \frac{1.5}{3.5} = \frac{3}{7}$

then $\frac{\overline{CE}}{\overline{CE'}} = \frac{\overline{EQ}}{\overline{E'F}} = \frac{\overline{CQ}}{\overline{CF}} = \frac{3}{7}$ (i)

Also $\frac{\overline{E'F}}{\overline{EB}} = \frac{1}{3}$, $\frac{\overline{EQ}}{\overline{E'F}} = \frac{3}{7}$ then $\frac{\overline{EQ}}{\overline{EB}} = \frac{1}{7}$ (ii)

By i and ii, $\overline{CQ} = \overline{QR} = 3\overline{RF}$

Similarly, $\overline{AR} = \overline{RP} = 3\overline{PD}$, 3EQ = QP = PB

Since $\overline{AR} = \overline{RP}$, ∠F′AR= ∠QPR, ∠ARF = ∠QRP

ΔQRP ≅ ΔF'RA

Since $\overline{EQ} = \frac{1}{7}\overline{EB}$, $\overline{CE} = \frac{1}{3}\overline{CA}$

Then SΔCEQ = $\frac{1}{3} \cdot \frac{1}{7}$ SΔABC (S is acronym for Area of a geometric figure)

Also SΔARF'= $\frac{1}{3}$CF' · h= $\frac{1}{3}$SΔACF'

Since $\frac{\overline{CE}}{\overline{AC}} = \frac{\overline{CQ}}{\overline{CF'}} = \frac{1}{3}$, SΔCEQ = $\frac{1}{9}$ SΔCAF'

So that SΔARF' = $\frac{1}{3}$ΔACF' = $\frac{1}{3} \cdot 9$ SΔCEQ = $\frac{1}{3} \cdot 9 \cdot \frac{1}{21}$ΔACB = $\frac{1}{7}$SΔABC

So that SΔQPR= $\frac{1}{7}$SΔABC

*John's Solution: Concept of Vectors*

John, on the other hand, used Dynamic Geometry software for construction but applied what he had learned in algebra to prove the conjecture. Asked why he decided to go with algebraic solution, John stated that, "I feel more confident in algebra than geometry". From John's point of view, Dynamic Geometry software did indeed play a significant role in the conjecturing process, but only a minor one in the solution process. In his solution, he used the concept of vectors to prove the conjecture as shown below:

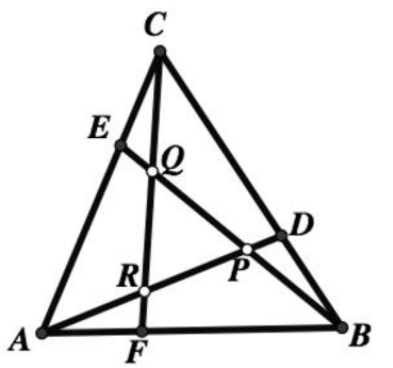


*Figure 7*: John's solution construction

Let $\triangle ABC$ be an arbitrary triangle. Let $D, E, F$ be points on the line segments

$$BD : BC = CE : CA = AF : AB = 1 : 3.$$

Let $\triangle PQR$ be formed by the construction of the line segments $AD, BE$ and $CF$.

Let $\overrightarrow{a}$ be the vector from $A$ to $B$, and let $\overrightarrow{b}$ be the vector from $A$ to $C$. Then the area of $\triangle ABC$ is $\frac{1}{2}\left|\overrightarrow{a} \times \overrightarrow{b}\right| = \frac{1}{2}\left|\overrightarrow{b} \times \overrightarrow{a}\right|$. Consider the vector $\overrightarrow{AR}$. We know that $\left|\overrightarrow{AR}\right| = s \cdot \left|\overrightarrow{AD}\right|$ for some $s \in [0,1]$ and that $\left|\overrightarrow{FR}\right| = t \cdot \left|\overrightarrow{FC}\right|$ for some $t \in [0,1]$.

Now

$$\overrightarrow{AD} = \overrightarrow{a} - \frac{1}{3}(\overrightarrow{a} - \overrightarrow{b}) = \frac{1}{3}(2\overrightarrow{a} + \overrightarrow{b}),$$

and

$$\overrightarrow{FC} = \overrightarrow{b} - \frac{1}{3}\overrightarrow{a} = \frac{1}{3}(3\overrightarrow{b} + \overrightarrow{a}).$$

Thus, we can write $\left|\overrightarrow{AR}\right| = \frac{s}{3}\left|2\overrightarrow{a} + \overrightarrow{b}\right|$ and $\left|\overrightarrow{FR}\right| = \frac{t}{3}\left|3\overrightarrow{b} - \overrightarrow{a}\right|$.

However, we may rewrite $\overrightarrow{AR} = \overrightarrow{AF} - \overrightarrow{RF}$, so

$$\begin{aligned}\left|\overrightarrow{AR}\right| &= \frac{1}{3}|\overrightarrow{a}| - \frac{t}{3}\left|3\overrightarrow{b} - \overrightarrow{a}\right| \\ &= \frac{1-t}{3}|\overrightarrow{a}| - t\left|\overrightarrow{b}\right| \\ &= \frac{2s}{3}|\overrightarrow{a}| - \frac{s}{3}\left|\overrightarrow{b}\right|.\end{aligned}$$

Thus, $t = \frac{s}{3}$, and $2s = 1 - t = 1 - \frac{s}{3}$. Solving, we obtain $s = \frac{3}{7}$ and $t = \frac{1}{7}$.

Now, $\overrightarrow{AR}$ is $\frac{3}{7}$ the length of $\overrightarrow{AD}$, and $\overrightarrow{RF}$ is $\frac{1}{7}$ the length of $\overrightarrow{FC}$. Since this ratio is independent of which side of triangle we choose, we also have that $\overrightarrow{PD}$ is $\frac{1}{7}$ the length of $\overrightarrow{AD}$, and $\overrightarrow{QC}$ is $\frac{3}{7}$ the length of $\overrightarrow{FC}$.

Let $\overrightarrow{x}$ be the vector from $R$ to $P$, and let $\overrightarrow{y}$ be the vector from $R$ to $Q$. Then $\overrightarrow{x}$ and $\overrightarrow{y}$ are $\frac{3}{7}$ the length of $\overrightarrow{AD}$ and $\overrightarrow{FC}$, respectively.

Now the area of $\triangle PQR$ is:

$$\begin{aligned}\frac{1}{2}|\overrightarrow{x} \times \overrightarrow{y}| &= \frac{1}{2}\left|\frac{s}{3}(2\overrightarrow{a} + \overrightarrow{b}) \times t(3\overrightarrow{b} - \overrightarrow{a})\right| = \frac{1}{2}\left|\frac{1}{7}(2\overrightarrow{a} + \overrightarrow{b}) \times \frac{1}{7}( \right. \\ &= \frac{1}{98}\left|(2\overrightarrow{a} + \overrightarrow{b}) \times (3\overrightarrow{b} - \overrightarrow{a})\right| \\ &= \frac{1}{98}\left|(6\overrightarrow{a} \times \overrightarrow{b}) + (2\overrightarrow{a} \times -\overrightarrow{a}) + (\overrightarrow{b} \times 3\overrightarrow{b}) + (\overrightarrow{b} \times - \right. \\ &= \frac{1}{98}\left|(6\overrightarrow{a} \times \overrightarrow{b}) + (\overrightarrow{b} \times -\overrightarrow{a})\right| \\ &= \frac{1}{98}\left|7\overrightarrow{a} \times \overrightarrow{b}\right| = \frac{1}{14}\left|\overrightarrow{a} \times \overrightarrow{b}\right| = \frac{1}{7}\left(\frac{1}{2}\left|\overrightarrow{a} \times \overrightarrow{b}\right|\right),\end{aligned}$$

which is $\frac{1}{7}$ the area of $\triangle ABC$.

*Figure 8*: John's Solution

It is important to note that this method of solution, using vectors, is specialist mathematics and therefore for senior mathematics.

John's story demonstrates the need for teaching geometry in high school using methods and strategies that efficiently help develop student skills about visualisation, perspective, problem-solving, conjecturing, and deductive reasoning. Also, if well thought through, geometry instruction can be a particularly appropriate subject area for furthering student understanding of the notion of proof. Although John had been teaching geometry, he never thought of demanding proofs from his students, firstly because the students were not required to do so, and secondly, because he was not comfortable teaching them that concept. With that said, teachers should not only teach the concepts they are comfortable in, the training that they are provide with in college should enable them to teach comfortably in the area of expertise. The initial conjecture that John made also indicated that he lacked visualisation skills since there was no way that ΔPQR could have been $\frac{1}{3}$ of triangle ΔABC.

*Mary's Solution: Construction Strategy*

For the case of Mary, she incorporated Dynamic Geometry software construction to her proof as illustrated below.

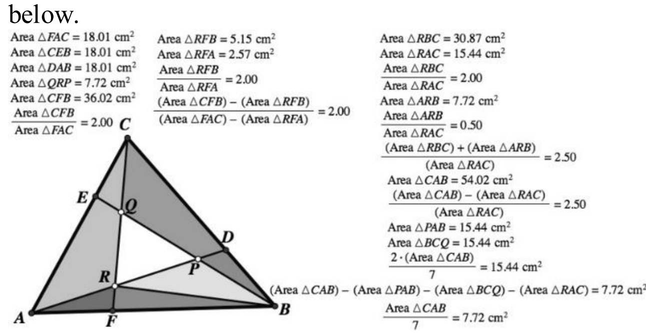


*Figure 9*: Mary's proof construction

After the construction, Mary proceeded with the solution below:

$\dfrac{Area\ \Delta CFB}{Area\ \Delta FAC} = 2$, similarly, $\dfrac{Area\ \Delta RFB}{Area\ \Delta RFA} = 2$, because in both cases the heights are the same length and the bases are at a 2:1 ratio (i)

From this, we have:

$$\frac{(Area\ \Delta CFB) - (Area \Delta RFB)}{(Area\ \Delta FAC) - (Area\ \Delta RFA)} = 2$$

Thus $\frac{Area\ \Delta RBC}{Area\ \Delta RAC} = 2,$ (ii)

which also gives us:

$\frac{Area\ \Delta ARB}{Area\ \Delta RAC} = \frac{1}{2},$ (iii)

Add (ii) and (iii), and we have

$\frac{(Area\ \Delta RBC) + (Area \Delta ARB)}{Area\ \Delta RAC} = \frac{5}{2}$ which is equivalent to

$\frac{(Area\ \Delta CAB) - (Area \Delta RAC)}{Area\ \Delta RAC}$

From this, we see that the area of $\Delta RAC = \frac{Area\ \Delta ABC}{3.5}$

$$Area \Delta PQR = Area\ (\Delta ABC - 3\Delta PBA) = \frac{0.5 Area\ \Delta ABC}{3.5}$$

$$Þ \quad Area \Delta PQR = \frac{Area \Delta ABC}{7}$$

Although Mary did demonstrate that her conjecture was true, there were still lingering questions about the solution. In her solution, she noted that that the height of ΔCFB and ΔFAC are at a 2:1 ratio, whereas it was the bases that were at that ratio while the height kept constant. The same was implied for the case of ΔRFB and ΔRFA. As above, it was not made clear why $\frac{Area\ \Delta ARB}{Area\ \Delta RAC} = \frac{1}{2}$ in her discussion. Next, in stage II, Mary also did not explain 3.PBA in the equation

$$Area \Delta PQR = Area\ (\Delta ABC - 3\Delta PBA) = \frac{0.5 Area\ \Delta ABC}{3.5}$$

Incorporating technology, in this case, dynamic geometry software, when teaching can help facilitate teachers/student efforts to construct, explore, conjecture and possibly prove. Although using paper and pencil may help students to investigate the activity presented; the dynamic software, on the other hand, enables teachers/students to investigate different cases of constructed figures and strategies that create an environment to conjecture, and possibly prove the conjecture. It is essential to provide the opportunity for teachers to use technology to realise its importance in the classroom regarding its suitability to discovery learning and being able to investigate mathematical problems that they could have not easily to examine using paper and pencil to construct, explore conjecture and possibly prove. Making it easier for teachers to engage with technology to support worthwhile activities, can have a great impact on teacher learning. Teachers need time to learn all features that are useful when teaching in a dynamic environment and have access to worthwhile activities to use, as Leikin (2015) notes that "the majority of teachers of mathematics in school nowadays do not have personal experience in learning mathematics through mathematical investigations, while many teachers have limited experience in the use of dynamic software for mathematical investigations" (p. 375). The dynamic nature of the software and measuring capability feature that the software has makes it possible to investigate several construction scenarios in real time (Trung, 2014).

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