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A Class of Delay Differential Inverse Variational Inequalities

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Abstract

This paper is focused on a class of delay differential inverse variational inequality composed of a system of delay differential equations and inverse variational inequalities. Also, the existence theorem of Carathéodory weak solution for delay differential inverse variational inequalities is established under suitable conditions. Furthermore, an algorithm for solving delay differential inverse variational inequalities is shown, and the convergence of the algorithm is given. Finally, a numerical example is given to simulate the effectiveness of the algorithm.

Keywords: Delay differential inverse variational inequalities; Variational inequalities; Carathéodory weak solution; Existence; Algorithm

1 Introduction

As is known to all that variational inequality is a very important non-linear problem which is widely used to solve optimization, programming problem, management science and economic equilibrium (see, for example, [1-4]). However, there are many parameters in reality changing with time, which the static variational inequality cannot be solved. Pang and Stewart [5] first introduced a class of differential variational inequalities (DVI for short) composed of a class of ordinary differential equations and variational inequalities in finite-dimensional Euclidean spaces in order to solve the brand new mathematical crossover problem, and the numerical solution of DVI was obtained by Euler time-stepping procedure. For some related work, we refer to the references [5-13]. Li et al [14] generalized the results mentioned above to differential mixed variational inequalities. However, there were multiple solutions for each step under this condition. A unique solution was guaranteed by the regularized time-stepping method in a Hilbert space. Very recently, Wang et al [15] studied the existence for DVI with relaxing the convexity condition.

DVI provides a powerful modeling paradigm for many applied problems in which dynamics, inequalities, and discontinuities are present; examples of such problems include constrained time-dependent physical systems with unilateral constrains, differential Nash games, and hybrid engineering systems with variable structures. Friesz [16] studied the differential Cournot-Nash game describing dynamic oligopolistic network competition via a DVI involving both control and state variables. Also, Raghunathan [17] considered parameter estimation in metabolic flux balance models for batch fermentation by using DVI.

Taking time delays often arisen in reality into account, Wang et al [18] introduced and studied a class of delay differential variational inequalities (DDVI for short) which is composed of a class of delay differential equation and variational inequalities. In this article, sufficient conditions for the existence of Carathéodory`s weak solutions are obtained. Also, a time-stepping method and its convergence analysis to find Carathéodory`s weak solutions are given. The results presented in this paper can be used to consider dynamic human migration and vector optimization constrained by delay differential equation.

In addition, He et al [19,20] first introduced and studied the inverse variational inequalities (IVI for short). Through the analysis of economics, management science and transportation, they pointed out that, among them, the problems of lower-level decision making can be formulated as a class of variational inequalities, but the problems of upper management usually can be formulated as a class of IVI. At the same time, a self-adaptive correction method for solving 'black-box' monotone IVI was given and its convergence was proved. Furthermore, He et al [21] advanced a method based on proximal algorithm for solving a class of constrained 'black-box' IVI. He and Liu [22] put forward two projection-based methods for solving IVI. Zou et al [23] proposed neural networks for solving IVI. Yang [24] presented a dynamic power price problem and characterized the optimal regulatory price as the solution of a IVI. Scrimali [25] proposed a problem of the time-dependent spatial price equilibrium and formulated an evolutionary IVI. Barbagallo and Mauro [26] studied the behavior of control policies for a dynamic oligopolistic market equilibrium and defined the optimal regulatory tax by using a IVI.

For solving the time-dependent problems, it is necessary to consider an ordinary differential equation. Therefore, Li et al [27] first studied and introduced a class of differential inverse variational inequalities (DIVI for short) in finite dimensional Euclidean spaces. The existence of Carathéodory weak solutions for DIVI was established. Moreover, they applied the DIVI for solving the time-dependent spatial price equilibrium control problem.

On this basis, taking the time delays into account, we introduce and study a class of delay differential inverse variational inequalities (DDIVI for short) composed of a class of delay differential equations and inverse variational inequalities.

The remaining part of this paper is organized as follows. In section 2, we present some preliminaries. In section 3, we establish sufficient conditions for the existence theorem of Carathéodory weak solutions of DDIVI. In section 4, we provide an Euler time-stepping method for solving the DDIVI and show the convergence analysis of the algorithm. Finally, we give a numerical simulation to prove the validity of the algorithm for solving the DDIVI.

2 Preliminaries

In this article, we introduce and study the following delay differential inverse variational inequality (DDIVI for short) composed of a class of delay differential equations and inverse variational inequalities:

Find$x: [t_0, t_0+T] \to \mathbb{R}^m$and $u: [t_0, t_0+T] \to \mathbb{R}^n$, such that DDIVI(1) holds,

$$\begin{cases} \dot{x}(t) = f(x(t-\tau), u(t)), & \text{for almost all } t \in [t_0, t_0+T], \\ G(x(t), u(t)) \in K \subset \mathbb{R}^n, \quad \langle v - G(x(t), u(t)), u(t) \rangle \geq 0, & \forall v \in K \subset \mathbb{R}^n, \\ x(t_0+s) = \varphi(s), & \forall s \in [-\tau, 0]. \end{cases}$$

where $\dot{x}(t):\ = \frac{dx}{dt}$, denotes the time-derivative of a function x(t), $t_0$denotes an initial time, and $\tau > 0$, $f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $G: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, and $\varphi: [-\tau, 0] \to \mathbb{R}^m$. are three given mappings.

Let SOL($\mathbb{R}^n, G(x,\cdot)$) denote the solution set of the following inverse variational inequality, which is to find$u \in \mathbb{R}^n$such that

$$G(x(t), u(t)) \in K \subset \mathbb{R}^n, \quad \langle v - G(x(t), u(t)), u(t) \rangle \geq 0, \forall v \in K \subset \mathbb{R}^n.$$

The pair of (x, u) is called a Carathéodory weak solution of DDIVI(1) if and only if x is an absolutely continuous function on $[t_0, t_0+T]$. Also, for almost all $t \in [t_0, t_0+T]$, it satisfies the following delay differential equation:

$$\begin{cases} \dot{x}(t) = f(x(t-\tau), u(t)), \\ x(t_0+s) = \varphi(s), \quad \forall s \in [-\tau, 0]. \end{cases}$$

where $u \in L^2([t_0, t_0+T], R^n)$, and $u(t) \in$ SOL($R^n, G(x(t),\cdot)$). Thus, the set of all Carathéodory weak solutions (x, u) for the initial-value DDIVI(1) is denoted by SOL(DDIVI(1)).

Definition 2.1 Let Y, Z be two Banach spaces. $F: Y \to Z$ is a strong continuous mapping if and only if, for any $\{x_n\} \subset Y$, when x_n weakly converges to x, for any$F$, there exists

$$F(x_n) \to F(x).$$

Lemma 2.1(The Schauder Fixed Point Theorem) [28] Let C be a nonempty, convex and compact subset of Banach space and $T: C \to C$ be a continuous mapping. Then the continuous mapping T has a fixed point in C.

Lemma 2.2[18] If f is continuous on $\mathbb{R}^m \times \mathbb{R}^n$ and G is continuous on $\mathbb{R}^m \times \mathbb{R}^n$. Then the pair (x, u) is a Carathéodory weak solution of DDIVI(1) if and only if the following three relations hold:

(i) for any $t_0 \leq t_1 \leq t_2 \leq t_0+T$,

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(x(s-\tau), u(s))\, ds.$$

(ii) for all $v \in L^2([t_0, t_0+T], K)$,

$$\int_{t_0}^{t_0+T} \langle v - G(x(t), u(t)), u(t) \rangle dt \geq 0.$$

(iii) the initial condition $x(t_0+s) = \varphi(s), \quad \forall s \in [-\tau, 0]$.

Proof The absolute continuity of x implies, for almost all $t \in ([t_0, t_0+T])$, the equivalence between

$$\dot{x}(t) = f(x(t-\tau), u(t)).$$

and

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s-\tau), u(s) ds.$$

When $u \in L^2([t_0, t_0+T], R^n)$, we can obtain that, for almost all $t \in ([t_0, t_0+T])$,

$$u \in SOL(\mathbb{R}^n, G(x(t),\cdot)).$$

is equivalent to

$$\int_{t_0}^{t_0+T} \langle v - G(x(t), u(t)), u(t) \rangle dt \geq 0.$$

for all $v \in L^2([t_0, t_0+T], K)$.

Lemma 2.3(Arzelá-Ascoli Theorem)[29] Let $S \subset C(K, X)$ be compact and let y^∞ be a sequence of continuous functions defined on $S \subset C(K, X)$. If $\{y^\infty\}$ is uniformly bounded and equicontinuous on $S \subset C(K, X)$ then there exists a subsequence $\{y^k\}$ that converges uniformly to a function $y \in S$.

Lemma 2.4(Lebesgue`s Dominated Convergence Theorem)[30] Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, that $f_n \to f$ pointwise almost everywhere as $n \to \infty$, and that $|f_n| \le g$ for all n, where g is integrable. Then f is integrable, and

(i) $\lim\limits_{n\to\infty} \int_E |f_n(x) - f(x)| dx = 0$,

(ii) $\lim\limits_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$.

Lemma 2.5(The Banach-Alaoglu Theorem) [31] Let X be a Banach space. Then the unit ball $\{f \in X' | \|f\| \le 1\}$ of X' is compact in the weak* topology.

3 Existence of the Solution of DDIVI

Let $C([t_0-\tau, t_0+T-\tau], \mathbb{R}^m)$denote the set of all the continuous functions from $[t_0-\tau, t_0+T-\tau]$ into $\mathbb{R}^m$, and $M([t_0, t_0+T], \mathbb{R}^n)$ denote the set of all the measurable functions from $[t_0, t_0+T]$ into$\mathbb{R}^n$.

Assumption 3.1 Assume that there exists $\eta > 0$, such that $\|f\|_M \le \eta$, where $\|\cdot\|_M$ denotes the maximum norm on $\mathbb{R}^m \times \mathbb{R}^n$.

In order to establish the existence theorem of Carathéodory weak solution of DDIVI(1), we firstly define

$$\hat{\varphi}(t) = \begin{cases} \varphi(t-t_0), & t_0-\tau \le t \le t_0, \\ \varphi(0), & t \ge t_0. \end{cases} \tag{3-1}$$

Assume that $(x(t), u(t))$ is a Carathéodory weak solution of DDIVI(1). Define

$$y(t_0) = x(t_0+t) - \hat{\varphi}(t_0+t), \quad t \ge -\tau. \tag{3-2}$$

It follows from(3-1) and (3-2) that, for every $t \in [-\tau, 0]$,

$$\begin{aligned} y(t) &= x(t_0+t) - \hat{\varphi}(t_0+t) \\ &= x(t_0+t) - \varphi(t_0+t-t_0) \\ &= x(t_0+t) - \varphi(t) \\ &= 0. \end{aligned}$$

and for every $t \in [-\tau, 0]$,

$$\begin{aligned} y(t) &= x(t_0+t) - \hat{\varphi}(t_0+t) \\ &= x(t_0+t) - \varphi(0) \\ &= \int_{t_0}^{t_0+T} f(x(s-\tau), u(s))ds \\ &= \int_0^t f(x(s+t_0-\tau), u(s+t_0))ds \\ &= \int_0^t f(\hat{\varphi}(s+t_0-\tau) + y(s-\tau), u(s+t_0))ds. \end{aligned} \tag{3-3}$$

Define

$$A(T, \eta T) = \{y \in C([-\tau, T], \mathbb{R}^m) : y(0) = 0, \|y\|_M \le \eta T\}. \tag{3-4}$$

and

$$Ty = \begin{cases} \int_0^t f(\hat{\varphi}(s+t_0-\tau) + y(s-\tau), u(s+t_0))ds, & t \in [0, T], \\ 0, & t \in [-\tau, 0]. \end{cases} \tag{3-5}$$

Lemma 3.1 Let Assumption 3.1 holds, f be continuous on $\mathbb{R}^m \times \mathbb{R}^n$. Then for every $u \in M([t_0, t_0+T], \mathbb{R}^n)$, there exists a fixed point $y \in A(T, \eta T)$ such that $y = Ty$.

Furthermore, there exists x such that

$$\begin{cases} x(t) = \varphi(0) + \int_0^t f(x(s-\tau), u(s))ds, \\ x(t_0+s) = \varphi(s), \quad \forall s \in [-\tau, 0]. \end{cases} \tag{3-6}$$

Proof Let Assumption 3.1 holds, $\|f\|_M \le \eta$. Then for any $t_1, t_2 \in [0, T]$,

$$\|\mathrm{T}(\mathrm{y}(t_1) - T(\mathrm{y}(t_2)\|_M \le \eta |t_1 - t_1|,$$
$$\|\mathrm{T}(\mathrm{y}(t)\|_M \le \eta T.$$

Define

$$S = \{y \in C([-\tau, T], \mathbb{R}^m) : \|y(t_1) - y(t_2)\|_M \le \eta |t_1 - t_2|, \\ \|y\|_M \le \eta T, y(0) = 0\}.$$

By Lemma 2.3(Arzelá-Ascoli Theorem), it implies that S is compact.

Next, we prove that the mapping $T: A(T, \eta T) \to A(T, \eta T)$ is continuous. Assume that $\{y^k\} \subset A(T, \eta T)$ and $y^k \to y^\infty$ as $y^k \to y^\infty$. For every $S \in [0, T]$, the continuity of f implies that

$$f(\hat{\varphi}(s+t_0-\tau) + y^k(s-\tau), u(s+t_0)) \\ \to f(\hat{\varphi}(s+t_0-\tau) + y^\infty(s-\tau), u(s+t_0))ds.$$

Then Lemma 2.4(Lebesgue dominated convergence theorem) implies that

$$\int_0^t f(\hat{\varphi}(s+t_0-\tau) + y^k(s-\tau), u(s+t_0))ds \\ \to \int_0^t f(\hat{\varphi}(s+t_0-\tau) + y^\infty(s-\tau), u(s+t_0))ds.$$

for every $t \in [0, T]$ and therefore, T is continuous on $A(T, \eta T)$.

Since $A(T, \eta T)$ is a closed and convex subset, T is a continuous mapping of $A(T, \eta T)$ into S, and S is a compact subset $A(T, \eta T)$. It follows from Lemma 2.1 that there exists a fixed point $y \in A(T, \eta T)$ satisfying $y = Ty$, it follows that

$$y(t) = \begin{cases} \int_0^t f(\hat{\varphi}(s+t_0-\tau) + y(s-\tau), u(s+t_0))ds, & t \in [0, T], \\ 0, & t \in [-\tau, 0]. \end{cases}$$

Then (3-1) and (3-2) imply that

$$x(t_0+t) = \hat{\varphi}(t_0+t) + y(t) \\ = \hat{\varphi}(t_0+t) + \int_0^t f(x(s+t_0-\tau), u(s+t_0))ds, \quad \forall t \in [0, T].$$

and so

$$x(t) = \hat{\varphi}(0) + \int_{t_0}^t f(x(s-\tau), u(s))ds, \quad \forall t \in [t_0, t_0+T].$$

Moreover, (3-1) and (3-2) imply that

$$x(t_0+s) = \varphi(s), \quad \forall s \in [-\tau, 0].$$

Theorem 3.1 Let Assumption 3.1 hold, $f(\cdot,\cdot)$and $G(\cdot,\cdot)$are continuous on $\mathbb{R}^m$ with respect to the first variable and strongly continuous on $L^2([t_0, t_0+T], \mathbb{R}^n)$ with respect to the second variable. Then SOL(DDIVI(2-1)) is non-empty.

Proof Take a measurable and uniformly bonded function,$u_n \in M([t_0, t_0+T], \mathbb{R}^n)$. By Lemma 3.1, there exists x_n such that

$$\begin{cases} x_n(t) = \varphi(t_0) + \int_{t_0}^t f(x_n(s-\tau), u_n(s))ds, & \forall t \in [t_0, t_0+T], \\ x_n(t_0+s) = \varphi(s), & \forall s \in [-\tau, 0]. \end{cases} \tag{3-7}$$

Since $K \subset \mathbb{R}^n$ is a compact and convex set, and G is continuous on $\mathbb{R}^m \times \mathbb{R}^n$, and x_n is continuous on $[t_0, t_0 + T]$, it follows that there exists a measurable function $u_{n+1} \in \mathrm{SOL}(\mathbb{R}^n, G(x_n(t), \cdot))$, and for all $v \in L^2([t_0, t_0 + T], K)$,

$$\int_{t_0}^{t_0+T} \langle v(t) - G(x_n(t), u_{n+1}(t)), u_{n+1}(t) \rangle dt \geq 0. \quad (3\text{-}8)$$

Then we get two sequences $\{x_n\}$ and $\{u_n\}$, $(n = 1,2, \dots \dots)$.

Since a sequence $\{x_n\} \in C([-\tau, T], \mathbb{R}^m)$, $(n = 1,2, \dots \dots)$ and

$$\|x_n(t_1) - x_n(t_2)\|_M \leq \eta |t_1 - t_1|,$$

$$\|y\|_M \leq \eta \mathrm{T},$$

$$y(0) = 0,$$

for any $t_1, t_2 \in [0, T]$.

Then $\{x_n\}$, $(n = 1,2 \dots \dots)$ is equicontinuous and uniformly bonded. By Lemma 2.3(Arzelá-Ascoli Theorem), there exists a subsequence of $\{x_n\}$, which we denoted by $\{x_n\}$ again, such that $x_n \to x$ in the maximum norm.

By the uniform boundedness of $u_n \in M([t_0, t_0 + T], \mathbb{R}^n)$ and Lemma 2.5(Alaoglu`s Theorem), it follows that the sequence $\{u_n\}$ has a weak* limit $\hat{u}$. in $L^2([t_0, t_0 + T], \mathbb{R}^n)$. The reflexive Banach space $L^2([t_0, t_0 + T], \mathbb{R}^n)$ implies that weak* convergent sequences are also weakly convergent sequences.

By (3-7) and (3-8), in addition, $f(\cdot,\cdot)$and $G(\cdot,\cdot)$are continuous on $\mathbb{R}^m$with respect to the first variable and strongly continuous on $L^2([t_0, t_0 + T], \mathbb{R}^n)$ with respect to the second variable, it implies that

$$\begin{cases} \hat{x}(t) = \varphi(t_0) + \int_{t_0}^{t} f(\hat{x}(s-\tau), \hat{u}(s)) ds, & \forall t \in [t_0, t_0 + T], \\ \hat{x}(t_0 + s) = \varphi(s), & \forall s \in [-\tau, 0]. \end{cases} \quad (3\text{-}9)$$

Then, for all $v \in L^2([t_0, t_0 + T], K)$,

$$\int_{t_0}^{t_0+T} \langle v(t) - G(\hat{x}(t), \hat{u}(t)), \hat{u}(t) \rangle dt \geq 0. \quad (3\text{-}10)$$

Therefore, $(\hat{x}, \hat{u})$ is a Carathéodory weak solution of DDIVI(2-1).

4 An Algorithm for DDIVI

In this article, a time-stepping method is used to find a weak solution of DDIVI(2-1). Let $[r]$ denote the biggest integer which is less than or equal to r.

Algorithm 4.1

Step 0: It begins with the division of the time interval $[t_0 - \tau, t_0 + T]$ into $\left[\frac{T}{l}\right] - \left[\frac{-\tau}{l}\right]$ subintervals:

$$t_0 - \tau = t_{l,[\frac{-\tau}{l}]} < t_{l,[\frac{-\tau}{l}+1]} < \cdots < t_{l,-1} < t_0$$
$$= t_{l,0} < t_{l,1} < \cdots < t_{l,[\frac{T}{l}-1]} < t_{l,[\frac{T}{l}]} = t_0 + T.$$

where $l > 0$, and $\left(\left[\frac{T}{l}\right] - \left[\frac{-\tau}{l}\right]\right) \times l = T + \tau$, and $t_{l,i+1} = t_{l,i} + l$, where

$$i = \left[\frac{-\tau}{l}\right], \cdots, -1,0,1, \cdots, \left[\frac{T}{l}\right].$$

Step 1: When $\left[\frac{-\tau}{l}\right] \leq i < 0$, let $x^{l,i} = \varphi(t_{l,i})$, and compute $u = u^{l,i}$, which satisfies the following variational inequality,

$$G(x, u) \in K \subset \mathbb{R}^n, \quad \langle v - G(x^{l,i}, u), u \rangle \geq 0, \quad \forall v \in K \subset \mathbb{R}^n. \quad (4\text{-}1)$$

Step 2: Take $k = \left[\frac{\tau}{l}\right]$. Let

$$x^{l,i+1} = \begin{cases} x^{l,i} + lf(\varphi(t_{l,i-k}), u^{l,i}), & 0 < i \leq k, \\ x^{l,i} + lf(x^{l,i-k}, u^{l,i}), & i > k. \end{cases}$$

and let $u = u^{l,i+1}$ be the solution of the following variational inequality.

$$G(x, u) \in K \subset \mathbb{R}^n, \quad \langle v - G(x^{l,i+1}, u), u \rangle \geq 0, \quad \forall v \in K \subset \mathbb{R}^n. \quad (4\text{-}2)$$

By the recursion, for $i = \left[\frac{-\tau}{l}\right], \cdots, -1,0,1,2, \cdots, \left[\frac{T}{l} - 1\right], \left[\frac{T}{l}\right]$, the following two finite families of vectors are obtained:

$$\left\{x^{l,\left[\frac{-\tau}{l}\right]}, \cdots, x^{l,0}, \cdots, x^{l,\left[\frac{T}{l}\right]}\right\} \subset \mathbb{R}^m,$$
$$\left\{u^{l,\left[\frac{-\tau}{l}\right]}, \cdots, u^{l,0}, \cdots, u^{l,\left[\frac{T}{l}\right]}\right\} \subset \mathbb{R}^n.$$

Finally, let

$$x^l(t) := x^{l,i} + \frac{t - t_{l,i}}{l}(x^{l,i+1} - x^{l,i}), \quad \forall t \in [t_{l,i}, t_{l,i+1}],$$
$$u^l(t) := u^{l,i} + \frac{t - t_{l,i}}{l}(u^{l,i+1} - u^{l,i}), \quad \forall t \in [t_{l,i}, t_{l,i+1}]. \quad (4\text{-}3)$$

Theorem 4.1 Let Assumption 3.1 hold. Then there exists a sequence $\{l_m\} \downarrow 0$ such that $x^{l_m} \to x$ and $u^{l_m} \to u$ in $L^2([t_0, t_0 + T], \mathbb{R}^n)$, where x^{l_m} and u^{l_m} are defined by (4-3).

Furthermore, assume that $f(\cdot,\cdot)$and $G(\cdot,\cdot)$are continuous on $\mathbb{R}^m$ with respect to the first variable and strongly continuous in $L^2([t_0, t_0 + T], \mathbb{R}^n)$ with respect to the second variable. Then all limits of (x, u) are weak solutions of DDIVI(2-1).

Proof Since

$$x^{l_m,i+1} = \begin{cases} x^{l_m,i} + l_m f(\varphi(t_{l,i-k}), u^{l_m,i}), & 0 < i \leq k, \\ x^{l_m,i} + l_m f(x^{l_m,i-k}, u^{l_m,i}), & i > k. \end{cases}$$

and

$$\|f\|_m \leq \eta,$$

which follows that

$$\left|x^{l_m,i+1} - x^{l_m,i}\right| \leq \eta l_m.$$

This implies that $\{x^{l_m}(t)\}$ is equicontinuous and uniformly bounded. By Lemma 2.3(Arzelá-Ascoli Theorem), there exists a sequence $\{l_m\} \downarrow 0$ such that $\{x^{l_m}\}$ converges to a function x in the maximum norm.

By the uniform boundedness of the sequence $\{u^{l_m}\}$ and Lemma 2.5(Alaoglu`s Theorem), it follows that $\{u^{l_m}\}$ is a weak* convergent sequence in $L^2([t_0, t_0 + T], \mathbb{R}^n)$. The reflexive Banach space $L^2([t_0, t_0 + T], \mathbb{R}^n)$ implies that weak* convergent sequences are also weakly convergent sequences.

Actually,

$$x^{l_m,i+1} - x^{l_m,i} = \begin{cases} l_m + f\big(\varphi(t_{l,i-k}), u^{l_m,i}\big), & 0 < i \leq k, \\ l_m + f(x^{l_m,i-k}, u^{l_m,i}), & i > k \end{cases}$$

$$= \int_{t_{l_m,i}}^{t_{l_m,i+1}} f(x^{l_m}(s-\tau), u^{l_m}(s)) ds + O(l_m).$$

Then for any $0 \leq t_1 \leq t_2 \leq T$, there exists

$$x^{l_m}(t_2) - x^{l_m}(t_1) = \int_{t_1}^{t_2} f(x^{l_m}(s-\tau), u^{l_m}(s)) ds + O(l_m).$$

It follows that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(x(s-\tau), u(s)) ds. \quad (4\text{-}4)$$

On the other hand, for every $v \in L^2([t_0, t_0 + T], K)$, we have

$$\left| \int_{t_0}^{t_0+T} \langle v(t) - G\big(x^{l_m}(t), u^{l_m}(t)\big), u^{l_m}(t) \rangle dt - \int_{t_0}^{t_0+T} \langle v(t) - G\big(x(t), u(t)\big), u(t) \rangle dt \right|$$
$$\leq \left| \int_{t_0}^{t_0+T} \langle v(t) - G\big(x^{l_m}(t), u^{l_m}(t)\big), u^{l_m}(t) \rangle dt - \int_{t_0}^{t_0+T} \langle v(t) - G\big(x(t), u(t)\big), u^{l_m}(t) \rangle dt \right|$$
$$+ \left| \int_{t_0}^{t_0+T} \langle v(t) - G\big(x(t), u(t)\big), u^{l_m}(t) \rangle dt - \int_{t_0}^{t_0+T} \langle v(t) - G\big(x(t), u(t)\big), u(t) \rangle dt \right|$$

It follows that as $l_m \to 0$,

$$\left| \int_{t_0}^{t_0+T} \langle v(t) - G(x^{l_m}(t), u^{l_m}(t)), u^{l_m}(t) \rangle dt - \int_{t_0}^{t_0+T} \langle v(t) - G(x(t), u(t)), u(t) \rangle dt \right| \to 0. \quad (4\text{-}5)$$

Then

$$\int_{t_0}^{t_0+T} \langle v(t) - G(x(t), u(t)), u(t) \rangle dt \geq 0. \quad (4\text{-}6)$$

$x^{l_m} \to x$ implies that

$$x(t_0 - s) = \varphi(s), \quad \forall s \in [-\tau, 0]. \quad (4\text{-}7)$$

Therefore, by the formula (4-4), (4-6) and (4-7), we obtain $(x, u) \in SOL(\text{DDIVI}(2-1))$.

5 Numerical Experiment

In Section 5, a numerical example is given to verify the effectiveness of the algorithm introduced in Section 4.

Example 5.1 Let

$$\begin{cases} f(x(t-\tau), u(t)) = x(t-0.4)u(t), & \varphi(t) = t^2, \\ G(x(t), u(t)) = x(t) - 0.1u(t), & K = [0,4]. \end{cases}$$

For every $t \in [-0.4, 3]$,

$$\begin{cases} \dot{x}(t) = x(t-0.4) \cdot u(t), & \forall t \in [0,3], \\ (x(t) - 0.1u(t)) \in [0,4], \quad \langle v - x(t) + 0.1 \cdot u(t), u(t) \rangle \geq 0, & \forall v \in [0,4], \\ x(t) = t^2, & \forall t \in [-0.4, 0]. \end{cases} \quad (5\text{-}1)$$

In the following, we will use the Euler time-stepping method given in Section 4 and show the specific iterations together with Example 5.1.

Algorithm 5.1

Step 0: Divide the time interval $[-0.4, 3]$ into small intervals with each of length $l = 0.05$.

$$t_{l,-8} = -0.4 < t_{l,-7} = -0.35 < t_{l,-6} = -0.3 < \cdots < t_{l,-1} = -0.05$$
$$< t_{l,0} = 0 < t_{l,1} = 0.05 < \cdots < t_{l,58} = 2.9 < t_{l,59} = 2.95 < t_{l,60} = 3.$$

Step 1: Let $x^{l,-8} = \varphi(t_{l,-8}) = 0.16$. Compute $u = u^{l,-8}$, which satisfies the following variational inequality,

$$(x^{l,0} + 0.1 \cdot u) \in [0,4], \quad \langle v - x^{l,0} + 0.1 \cdot u, u \rangle \geq 0, \quad \forall v \in [0,4].$$

Step 2: Take $k = \left[\frac{\tau}{l}\right] = \frac{0.4}{0.05} = 8$. When $-8 < i \leq 0$, let

$$x^{l,i+1} = x^{l,i} + l \cdot f(\varphi(t_{l,i}), u^{l,i})$$
$$= x^{l,i} + l \cdot (t_{l,i})^2 \cdot u^{l,i}.$$

Compute $u = u^{l,i+1}$, which satisfies the following variational inequality,

$$(x^{l,i+1} + 0.1 \cdot u) \in [0,4], \quad \langle v - x^{l,i+1} + 0.1 \cdot u, u \rangle \geq 0, \quad \forall v \in [0,4]. \quad (5\text{-}2)$$

Step 3: When $0 < i \leq k$, let

$$x^{l,i+1} = x^{l,i} + l \cdot f(\varphi(t_{l,i-k}), u^{l,i})$$
$$= x^{l,i} + l \cdot (t_{l,i-k})^2 \cdot u^{l,i}.$$

Compute $u = u^{l,i+1}$, which satisfies the following variational inequality,

$$(x^{l,i+1} + 0.1 \cdot u) \in [0,4], \quad \langle v - x^{l,i+1} + 0.1 \cdot u, u \rangle \geq 0, \quad \forall v \in [0,4]. \quad (5\text{-}3)$$

Step 4: When $k < i$, let

$$x^{l,i+1} = x^{l,i} + l \cdot f(x^{l,i}, u^{l,i})$$
$$= x^{l,i} + l \cdot x^{l,i-k} \cdot u^{l,i}.$$

Compute $u = u^{l,i+1}$ which satisfies the following variational inequality,

$$(x^{l,i+1} + 0.1 \cdot u,) \in [0,4], \quad \langle v - x^{l,i+1} + 0.1 \cdot u, u \rangle \geq 0, \quad \forall v \in [0,4]. \quad (5\text{-}4)$$

6 conclusions

In this article, we introduced and studied a class of delay differential inverse variational inequality consisted of inverse variational inequality and a system of delay differential equation. We gave the sufficient conditions to prove the existence of Carathéodory weak solution. Also, an algorithm was given to find the Carathéodory weak solution and the convergence was shown.

Furthermore, it is essential to take advantage of the conclusions of this article to consider the problems of upper management constrained by a system of delay differential equation and time-dependent spatial price equilibrium control with time delay.

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